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T_0 Separation in Axiomatic Quantum Mechanics[†]

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Using the physical duality between states and properties, Aerts *et al.* obtained a "lattice" representation for all closure spaces, through state property systems. In this paper I discuss the equivalence of 'state determination' for state property systems with T_0 separation for closure spaces. I also provide a link with well-known lattice representations of closure spaces, through some results of Erné.

1. DUALITY OF STATES AND PROPERTIES

In the Geneva–Brussels approach to the foundations of physics (Piron, 1976, 1989, 1990; Aerts, 1982, 1983; Moore, 1999) an entity is described by its 'states' and its 'properties.' *The* physical relation between states and properties is 'actuality.' So, I presuppose the existence of a relation I between the set $\mathcal{L}$ of properties and the set Σ of states,[3] where for $a \in \mathcal{L}$ and $p \in \Sigma$, aIp is interpreted as "property a is actual in state p." As with any relation, I induces a closure on Σ and on $\mathcal{L}$. This is a result due to Birkhoff (1967, Chapter 5, §7). Indeed, if for $A \subset \Sigma$ and $B \subset \mathcal{L}$ one puts

$$A^{+} = \{a \in \mathcal{L} | aIp \ \forall p \in A\} \tag{1}$$

$$B^{*} = \{p \in \Sigma | aIp \ \forall a \in B\} \tag{2}$$

then $\mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$, $A \mapsto A^{\dagger *}$, and $\mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$, $B \mapsto B^{*\dagger}$, are closure operators. Moreover, Birkhoff proves that the mappings $A \mapsto A^{+}$ and $B \mapsto B^{*}$ define dual isomorphisms between the complete lattices of closed subsets of Σ and $\mathcal{L}$.

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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[3] In the language of formal concept analysis (Ganter and Wille, 1999), ($\mathcal{L}$, Σ, I) is a 'context.'

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The relation I encodes the "physical duality of states and properties" (Moore, 1999). Using I, one can define a preorder on $\mathscr{L}$:

$$a < b \Leftrightarrow a^* \subset b^* \tag{3}$$

with a slight abuse of notation. It is of course "physical implication with respect to actuality" (Moore, 1999). After identification of mutually implying properties, $\mathscr{L}$ becomes a poset. It is then operationally justified (a key result in the Geneva School) to demand that $\mathscr{L}$ is a complete lattice in which the meet is "physical conjunction with respect to actuality" (Moore, 1999):

$$(\wedge_i a_i) I p \Leftrightarrow a_i I p \quad \forall_i \tag{4}$$

which is read as "$\wedge_i a_i$ is actual iff every a_i is actual." This is equivalent to demanding every closed subset of Σ be of the form a^* for some property $a \in \mathscr{L}$. Indeed, let $F = F^{\dagger *}$ be a closed subset of Σ. From (4) it follows that $(\wedge F^{\dagger})^* = \bigcap_{a \in F^{\dagger}} a^* = F^{\dagger *} = F$. Conversely, consider a family $a_i \in \mathscr{L}$. Then there exists $b \in \mathscr{L}$ such that $b^* = \bigcap_i a_i^*$. Consequently, $b = \wedge_i a_i$ and (4) holds. In the Geneva School, the mapping $a \mapsto a^*$, which is now a lattice isomorphism between $\mathscr{L}$ and the lattice of closed subsets of Σ, has been named the *Cartan map*.

Putting these ideas into one structure and writing $\xi(p)$ for $p^{\dagger}$ ($p \in \Sigma$), Aerts *et al.* (1999) defined state property systems. A triple $(\Sigma, \mathscr{L}, \xi)$ is a *state property system* (*sps*) if Σ is a set, $(\mathscr{L}, \wedge, <)$ is a complete lattice, and $\xi: \Sigma \to \mathscr{P}(\mathscr{L})$ is a function such that $\xi(p)$ never contains the universal lower bound 0 of $\mathscr{L}$ (0 is never actual) and

$$a < b \Leftrightarrow \text{if } a \in \xi(p) \text{ then } b \in \xi(p) \tag{5}$$

$$\wedge_i a_i \in \xi(p) \Leftrightarrow a_i \in \xi(p) \quad \forall_i \tag{6}$$

Obviously (5) and (6) are restatements of (3) and (4). Putting $s_\xi(p) = \wedge \xi(p)$, it is clear that $\xi(p) = [s_\xi(p), 1]$ for every $p \in \Sigma$ (1 is the maximum of $\mathscr{L}$). Evidently, $s_\xi(p)$ is the strongest (minimal) property which is actual in state p.

A couple (m, n) is a morphism $(\Sigma', \mathscr{L}', \xi') \to (\Sigma, \mathscr{L}, \xi)$ of sps's if $m: \Sigma' \to \Sigma$ and $n: \mathscr{L} \to \mathscr{L}'$ are maps such that for $p' \in \Sigma'$ and $a \in \mathscr{L}$

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{7}$$

For the physical idea behind this definition, I refer to Aerts *et al.* (1999). The category of sps's and their morphisms is denoted **SP**. The two functors given in (8) and (9) below establish an equivalence between **SP** and the category of closure spaces with Ø closed and continuous maps **Cls** (Aerts *et al.*, 1999).

$$F: \quad \mathbf{SP} \to \mathbf{Cls}, \qquad \begin{cases} (\Sigma, \mathscr{L}, \xi) \mapsto (\Sigma, \mathscr{F}_{\mathscr{L}}) \\ (m, n) \mapsto m \end{cases} \tag{8}$$

$$G: \quad \mathbf{Cls} \to \mathbf{SP}, \qquad \begin{cases} (\Sigma, \mathscr{F}) \mapsto (\Sigma, \mathscr{F}, \xi_{\mathscr{F}}) \\ m \mapsto (m, m^{-1}) \end{cases} \tag{9}$$

where $\mathscr{F}_{\mathscr{L}} = \{\{p \in \Sigma: a \in \xi(p)\} | a \in \mathscr{L}\}$ and $\xi_{\mathscr{F}}: \Sigma \to \mathscr{P}(\mathscr{F})$, $p \mapsto \{F \in \mathscr{F} | p \in F\}$. So, on the object side, F constructs one of the "Birkhoff polarity closures" for $I \subset \mathscr{L} \times \Sigma$ defined by aIp if $a \in \xi(p)$. Conversely, the object correspondence of G is presented in Aumann (1970). This equivalence provides a "lattice representation" for *all* closure spaces and, in this sense, generalizes Erné's (1984) lattice representation for T_0 closure spaces. As will be explained in Section 4, the latter representation is the cornerstone of Erné's general construction providing "all" lattice representable categories of closure spaces. In the same section I shall show how the above equivalence fits into this scheme.

In the final section I give a short discussion of T_0 separation for orthogonality spaces.

2. STATE DETERMINATION AND T_0 SEPARATION

Traditionally, in the Geneva–Brussels approach, the state p of an entity is identified with its actual properties, i.e., with $\xi(p)$. In this section I review the implications of this (physical) assumption on the equivalence of **SP** and **Cls**. Since they are straightforward, I omit the proofs, which can be found in Aerts *et al.* (1999).

We call an sps $(\Sigma, \mathscr{L}, \xi)$ *state determined* if ξ is injective.[4] In other words, $(\Sigma, \mathscr{L}, \xi)$ is state determined if every state $p \in \Sigma$ is determined by its actual properties $\xi(p)$. Let $\mathbf{SP}_0$ be the full subcategory of **SP**, with state-determined sps's as objects.

Lemma 1. Let $(\Sigma, \mathscr{L}, \xi)$ be an sps. The following are equivalent:

1. $(\Sigma, \mathscr{L}, \xi)$ is state determined.
2. $s_\xi: \Sigma \to \mathscr{L}, p \mapsto \wedge\xi(p)$, is injective.
3. $F(\Sigma, \mathscr{L}, \xi)$ is a T_0 closure space.

Conversely, a closure space $(\Sigma, \mathscr{F})$ is T_0 iff $G(\Sigma, \mathscr{F}) = (\Sigma, \mathscr{F}, \xi_{\mathscr{F}})$ is a state-determined sps.

Recall that a closure space $(\Sigma, \mathscr{F})$ is said to be T_0 if $cl(p) = cl(q) \Rightarrow p = q$ for $p, q \in \Sigma$, where for $A \subset \Sigma$, $cl(A) \doteqdot \cap\{F \in \mathscr{F} | A \subset F\}$.

[4] In formal concept analysis the corresponding contexts are called 'clarified.'

Let $\mathbf{Cls}_0$ be the full subcategory of **Cls** of T_0 closure spaces. The next proposition easily follows.

Proposition 1. The functors F and G of (8) and (9) restrict and corestrict to equivalence establishing functors between $\mathbf{SP}_0$ and $\mathbf{Cls}_0$.

3. STATES AS STRONGEST ACTUAL PROPERTIES

Let $(\Sigma, \mathscr{L}, \xi)$ be a state-determined sps. Then, by condition 2 of Lemma 1, a state $p \in \Sigma$ may be identified with $s_\xi(p) \in \mathscr{L}$, i.e., with the strongest property it makes actual. As a consequence, Σ can be embedded into $\mathscr{L}$ as an order-generating subset (see Lemma 2). This engenders another equivalence of categories (Proposition 2). The proofs can again be found in Aerts *et al.* (1999).

Lemma 2. Let $(\Sigma,\mathscr{L},\xi)$ be an sps. Then $0 \notin \Sigma^\xi \doteq s_\xi(\Sigma)$ is an order-generating subset of $\mathscr{L}$: for every a in $\mathscr{L}$,

$$a = \vee\{x \in \Sigma^\xi | x < a\} \tag{10}$$

A couple $(\Sigma, \mathscr{L})$ is a *based complete lattice* (*bcl*) if $\mathscr{L}$ is a complete lattice and $\Sigma \subset \mathscr{L}$ is an order-generating subset not containing 0. The previous lemma then becomes $(\Sigma, \mathscr{L}, \xi) \in \mathbf{SP} \Rightarrow (\Sigma^\xi, \mathscr{L})$ is a bcl.

Lemma 3. Let $(\Sigma,\mathscr{L})$ be a bcl. If we define

$$\xi: \Sigma \to \mathscr{P}(\mathscr{L}),\ p \mapsto [p,1] \tag{11}$$

then $(\Sigma, \mathscr{L}, \xi)$ is a state-determined sps.

To deal with the morphisms, I shall use Galois connections. I shall write n_* for the lower adjoint of a meet-preserving map n and f^* for the upper adjoint of a join-preserving f.

Consider two bcl's $(\Sigma', \mathscr{L}')$, $(\Sigma, \mathscr{L})$. A function $\bar{f}: \mathscr{L}' \to \mathscr{L}$ is a *morphism of bcl's* if f preserves joins and $f(\Sigma') \subset \Sigma$. The category of bcl's will be denoted $\mathbf{L}_0$.

Proposition 2. The following two functors establish an equivalence between $\mathbf{SP}_0$ and $\mathbf{L}_0$:

$$H: \quad \mathbf{SP}_0 \to \mathbf{L}_0, \qquad \begin{cases} (\Sigma, \mathscr{L}, \xi) \mapsto (\Sigma^\xi, \mathscr{L}) \\ (m, n) \mapsto n_* \end{cases} \tag{12}$$

$$K: \quad \mathbf{L}_0 \to \mathbf{SP}_0, \qquad \begin{cases} (\Sigma, \mathscr{L}) \mapsto (\Sigma, \mathscr{L}, \xi) \\ f \mapsto (f|_{\Sigma}^{\Sigma'}, f^*) \end{cases} \tag{13}$$

where ξ of (13) is given in (11).

The results above can be summarized in the following scheme, which can be read as a commutative diagram.

$$\begin{array}{ccccc} \mathbf{Cls} & \approx & \mathbf{SP} & & \\ \cup & & \cup & & \\ \mathbf{Cls}_0 & \approx & \mathbf{SP}_0 & \approx & \mathbf{L}_0 \end{array} \tag{14}$$

4. CONNECTION WITH ERNÉ'S RESULT

Erné (1984) gives a direct proof of the equivalence of $\mathbf{Cls}_0$ and $\mathbf{L}_0$. In fact, his functors are $H \circ G$ and $F \circ K$. Based on this equivalence, he gives a general construction for "lattice representable" (or *l-representable*, after Erné) categories of closure spaces. I give an outline of his result. Note that I shall change his definition of 'invariant selection' slightly: for Erné the empty set need not be closed.

Given a category $\mathbf{C}$ of closure spaces, i.e., a full and isoclosed subcategory of $\mathbf{Cls}$, Erné introduces the *l*-representing functor

$$T\colon \quad \mathbf{C} \to \mathbf{L}_\vee, \qquad \begin{cases} (\Sigma, \mathcal{F}) \mapsto (\mathcal{F}, \cap, \subset) \\ f \mapsto [F \mapsto cl(f(F))] \end{cases} \tag{15}$$

where $\mathbf{L}_\vee$ is the category of complete lattices with join-preserving maps. $\mathbf{C}$ is then called *l-representable* if a suitable corestriction of T yields an equivalence of categories. Erné gives many examples of such categories. Well known is the equivalence of the category of T_1 closure spaces and the category of complete atomistic lattices.

Next, let $\mathbb{L}$ be an isoclosed subclass of the class of complete lattices. An *invariant selection* S for $\mathbb{L}$ is a (class-theoretic) function assigning to each $\mathcal{L} \in \mathbb{L}$ a certain $0 \notin S(\mathcal{L}) \subset \mathcal{L}$, such that whenever ψ is a lattice isomorphism between $\mathbb{L}$-elements $\mathcal{L}$ and $\mathcal{L}'$, then $\psi(S(\mathcal{L})) = S(\mathcal{L}') = S(\psi(\mathcal{L}))$. Given an invariant selection S, Erné defines the isoclosed subcategory $\mathbf{L}_S$ of $L_\vee$ as follows. A complete lattice $\mathcal{L} \in \mathbb{L}$ is an object of $\mathbf{L}_S$ iff $S(\mathcal{L})$ is an order-generating subset of $\mathcal{L}$. An $\mathbf{L}_S$-morphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ is a join-preserving map, such that $\varphi(S(\mathcal{L})) \subset S(\mathcal{L}')$. Hence, $\mathbf{L}_S$ may be (and is) considered a full subcategory of $\mathbf{L}_0$.

He calls a closure space $(\Sigma, \mathcal{F})$ *S-complete* if it is T_0, $T(\Sigma, \mathcal{F}) \in \mathbb{L}$, and $S(T(\Sigma, \mathcal{F})) = \{cl(p) | p \in \Sigma\}$. The S-complete closure spaces form an isoclosed and full subcategory of $\mathbf{Cls}_0$, denoted $\mathbf{C}_S$. Finally, I state Erné's theorems.

Theorem 1. For every invariant selection S, the categories $\mathbf{C}_S$ and $\mathbf{L}_S$ are equivalent.

This equivalence is a restriction of the equivalence between $\mathbf{Cls}_0$ and $\mathbf{L}_0$. The next theorem says that all *l*-representable categories of closure spaces can be obtained this way.

Theorem 2. A category $\mathbf{C}$ of closure spaces is *l*-representable if and only if there exists an isoclosed class $\mathbb{L}$ of complete lattices and an invariant selection S for $\mathbb{L}$ such that $\mathbf{C} = \mathbf{C}_S$.

Given an isoclosed class $\mathbb{L}$ and an invariant selection S, I also introduce the full subcategory $\mathbf{SP}_S$ of $\mathbf{SP}$ of state property systems $(\Sigma,\mathcal{L},\xi)$ such that $F(\Sigma,\mathcal{L},\xi) = (\Sigma,\mathcal{F}_{\mathcal{L}})$ is in $\mathbf{C}_S$. Using that $F \circ G$ is the identical functor, it is obvious that F and G (co)restrict to an equivalence $\mathbf{C}_S \approx \mathbf{SP}_S$. Recall that the (Cartan) isomorphism κ between $\mathcal{L}$ and $\mathcal{F}_{\mathcal{L}}$ is given by $\kappa(a) = \{p \in \Sigma \mid a \in \xi(p)\}$. If $(\Sigma,\mathcal{L},\xi)$ belongs to $\mathbf{SP}_S$, then $\mathcal{F}_{\mathcal{L}} \in \mathbb{L}$ and $S(\mathcal{F}_{\mathcal{L}}) = S(\kappa(\mathcal{L})) = \{cl(p) \mid p \in \Sigma\}$. Therefore, since $\mathcal{L} \cong \mathcal{F}_{\mathcal{L}}$, $\mathcal{L} \in \mathbb{L}$ and $S(\mathcal{L}) = \kappa^{-1}(S(\kappa(\mathcal{L}))) = \Sigma^{\xi}$. It follows that $H(\Sigma,\mathcal{L},\xi)$ is in $\mathbf{L}_S \subset \mathbf{L}_0$. Using that $F \circ K$: $\mathbf{L}_S \to \mathbf{C}_S$ (this is one of Erné's functors in Theorem 1), it is now straightforward that H and K (co)restrict to an equivalence $\mathbf{SP}_S \approx \mathbf{L}_S$.

Summarizing, any *l*-representable category of closure spaces $\mathbf{C}$ fits, for a suitable invariant selection S, into the following scheme, which can be read as a commutative diagram and which shows how the equivalences of Sections 1–3 refine Erné's beautiful result:

$$\begin{array}{ccccc} \mathbf{Cls} & \approx & \mathbf{SP} & & \\ \cup & & \cup & & \\ \mathbf{Cls}_0 & \approx & \mathbf{SP}_0 & \approx & \mathbf{L}_0 \\ \cup & & \cup & & \cup \\ \mathbf{C} = \mathbf{C}_S & \approx & \mathbf{SP}_S & \approx & \mathbf{L}_S \end{array} \tag{16}$$

5. ORTHOGONALITY SPACES AND T_0 SEPARATION

Let Σ be a set and let $\perp$ be an antireflexive and symmetric relation on Σ. I then call $(\Sigma,\perp)$ a *pseudo orthogonality space (pos)*. If for $A \subset \Sigma$, $A^{\perp} = \{p \in \Sigma \mid p \perp a \ \forall\ a \in A\}$, *then* $\mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$, $A \mapsto A^{\perp\perp}$ is a closure operator such that $\emptyset^{\perp\perp} = \emptyset$. Moreover, $A \mapsto A^{\perp}$ is an orthocomplementation on the lattice of closed (biorthogonal) subsets (Birkhoff, 1967, Chapter 5, §7; Moore, 1995). Moore (1995) has proven that this closure is T_1 iff it separates points: $p \neq q \Rightarrow \exists r$: $p \perp r$, $q \not\perp r$; which is equivalent to $p \neq q \Rightarrow p^{\perp} \not\subset q^{\perp}$. The couple $(\Sigma, \perp)$ is then called a 'state space' or an 'orthogonality space' (Moore, 1995, 2000). T_0 separation can be characterized analogously.

Proposition 3. Let $\perp$ be a symmetric relation on Σ. The closure operator $\mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$, $A \mapsto A^{\perp\perp}$, is T_0 iff

$$p \neq q \Rightarrow p^{\perp} \neq q^{\perp} \tag{17}$$

Indeed, if (17) holds and $p^{\perp\perp} = q^{\perp\perp}$, then $p^{\perp\perp\perp} = p^{\perp} = q^{\perp}$, whence $p =$

q. Conversely, if $p^{\perp} = q^{\perp}$, then $p^{\perp\perp} = q^{\perp\perp}$, and so by T_0, $p = q$. In fact, the mathematics behind Lemma 1 and this Proposition are the same.

Consider two pos's $(\Sigma_1, \perp_1)$ and $(\Sigma_2, \perp_2)$ and the following three pos's:

1. $\Sigma_1 \times \Sigma_2$ with $(p_1, p_2) \perp_{\Pi} (q_1, q_2) \Leftrightarrow p_1 \perp_1 q_1$ and $p_2 \perp_2 q_2$.
2. $\Sigma_1 \dot{\cup} \Sigma_2$ with $p_i \perp_{\amalg} q_j \Leftrightarrow i \neq j$ or ($i = j$ and $p_i \perp_i q_i$) $(i, j \in \{1, 2\})$.
3. $\Sigma_1 \times \Sigma_2$ with $(p_1, p_2) \perp_{\oslash} (q_1, q_2) \Leftrightarrow p_1 \perp_1 q_1$ or $p_2 \perp_2 q_2$.

The closure associated to the first pos is the product (in **Cls**) of $(\Sigma_1, \cdot^{\perp_1\perp_1})$ and $(\Sigma_2, \cdot^{\perp_1\perp_1})$. The second pos generates the coproduct $(\Sigma_1, \cdot^{\perp_2\perp_2}) \amalg (\Sigma_2, \cdot^{\perp_2\perp_2})$. The last one is the 'separated product' introduced by Aerts (1982). For the first two it is well known that they are T_0 (T_1) iff Σ_1 and Σ_2 are. That this is also true for the separated product in the T_1 case was shown by Aerts (1982, Theorem 26).

Proposition 4. The separated product of Σ_1 and Σ_2 is T_0 iff Σ_1 and Σ_2 are.

Sufficiency can be proven as follows. Take $(p_1, p_2) \neq (q_1, q_2)$ and suppose that $p_1 \neq q_1$ and $p_1^{\perp} \ni r_1 \notin q_1^{\perp}$. Then $(p_1, p_2) \perp (r_1, q_2) \not\perp (q_1, q_2)$. For necessity one can assume Σ_1 and Σ_2 are nonempty. Consider $p_1 \neq q_1$ in Σ_1 and take $r_2 \in \Sigma_2$. Then $(p_1, r_2)^{\perp} \neq (q_1, r_2)^{\perp}$. Suppose there is a (t_1, t_2) such that $(p_1, r_2) \perp (t_1, t_2) \not\perp (q_1, r_2)$; then $p_1 \perp t_1 \not\perp q_2$.

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